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Abstract We study the existence and uniqueness of solutions of $\partial_t u - \Delta u + u^q = 0$ $(q > 1)$ in $\Omega \times (0,\infty)$ where $\Omega \subset \mathbb{R}^N$ is a domain with a compact boundary, subject to the conditions $u = f \geq 0$ on $\partial\Omega \times (0,\infty)$ and the initial condition $\lim_{t\to 0} u(x,t) = \infty$. By means of Brezis' theory of maximal monotone operators in Hilbert spaces, we construct a minimal solution when $f = 0$, whatever is the regularity of the boundary of the domain. When $\partial\Omega$ satisfies the parabolic Wiener criterion and f is continuous, we construct a maximal solution and prove that it is the unique solution which blows-up at $t = 0$.

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1 Introduction

Let Ω be a domain of $\mathbb{R}^N$ $(N \geq 1)$ with a compact boundary, $Q_\infty^\Omega = \Omega \times (0,\infty)$ and $q > 1$. This article deals with the question of the solvability of the following Cauchy-Dirichlet problem $\mathcal{P}^{\Omega,f}$

$$\left\{\begin{aligned} \partial_t u - \Delta u + |u|^{q-1}u = 0 \quad & \text{in } Q_\infty^\Omega \\ u = f \quad & \text{on } \partial\Omega \times (0,\infty) \\ \lim_{t\to 0} u(x,t) = \infty \qquad & \forall x \in \Omega. \end{aligned}\right. \tag{1.1}$$

If no assumption of regularity is made on $\partial\Omega$, the boundary data $u = f$ cannot be prescribed in sense of continuous functions. However, the case $f = 0$ can be treated if the vanishing condition on $\partial\Omega \times (0,\infty)$ is understood in the H_0^1 local sense. We construct a positive solution $\underline{u}_\Omega$ of (1.1) with $f = 0$ belonging to $C(0,\infty; H_0^1(\Omega) \cap L^{q+1}(\Omega))$ thanks to Brezis results of contractions semigroups generated by subdifferential of proper convex functions in Hilbert spaces. We can also consider an internal increasing approximation of Ω by smooth bounded domains Ω^n such that $\Omega = \cup_n \Omega^n$. For each of these domains, there exists a maximal solution $\overline{u}_{\Omega^n}$ of problem $\mathcal{P}^{\Omega^n,0}$. Furthermore the sequence $\{\overline{u}_{\Omega^n}\}$ is increasing. The limit function $u_\Omega := \lim_{n\to\infty} \overline{u}_{\Omega^n}$ is the natural candidate to be the minimal positive solution of a solution of $\mathcal{P}^{\Omega,0}$. We prove that $\underline{u}_\Omega = u_\Omega$. If $\partial\Omega$ satisfies the parabolic Wiener criterion [9], there truly exist solutions of $\mathcal{P}^{\Omega,0}$. We construct a maximal solution $\overline{u}_\Omega$ of this problem. Our main result is the following:

Theorem 1. *If $\partial\Omega$ is compact and satisfies the parabolic Wiener criterion, there holds*

$$\overline{u}_\Omega = \underline{u}_\Omega.$$

In the last section, we consider the full problem $\mathcal{P}^{\Omega,f}$. Under the same regularity and boundedness assumption on $\partial\Omega$ we construct a maximal solution $\overline{u}_{\Omega,f}$ and we prove

Theorem 2. *If $\partial\Omega$ is compact and satisfies the parabolic Wiener criterion, and if $f \in C(0,\infty;\partial\Omega)$ is nonnegative, $\overline{u}_{\Omega,f}$ is the only positive solution to problem $\mathcal{P}^{\Omega,f}$.*

These type of results are to be compared with the ones obtained by the same authors [1] in which paper the following problem is considered

$$\left\{\begin{array}{c} \partial_t u - \Delta u + |u|^{q-1}u = 0 \quad \text{in } Q_\infty^\Omega \\ \lim_{\text{dist}\,(x,\partial\Omega)\to 0} u(x,t) = \infty \quad \text{locally uniformly on } (0,\infty) \\ u(x,0) = f \qquad \forall x \in \Omega. \end{array}\right. \tag{1.2}$$

In the above mentioned paper, it is proved two types of existence and uniqueness result with $f \in L^1_{loc}(\Omega)$, $f \geq 0$: either if $\partial\Omega = \partial\overline{\Omega}^c$ and $1 < q < N/(N-2)$, or if $\partial\Omega$ is locally the graph of a continuous function and $q > 1$.

Our paper is organized as follows: 1- Introduction. 2- Minimal and maximal solutions. 3- Uniqueness of large solutions. 4- Bibliography.

2 Minimal and maximal solutions

Let $q > 1$ and Ω be a proper domain of $\mathbb{R}^N$, $N > 1$ with a non-empty compact boundary. We set $Q_\infty^\Omega = \Omega \times (0,\infty)$ and consider the following problem

$$\left\{\begin{array}{c} \partial_t u - \Delta u + u^q = 0 \quad \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0 \qquad \text{on } \partial\Omega \times (0,\infty). \end{array}\right. \tag{2.1}$$

If there is no regularity assumption on $\partial\Omega$, a natural way to consider the boundary condition is to impose $u(.,t) \in H^1_0(\Omega)$. The Hilbertian framework for this equation has been studied by Brezis in a key article [2] (see also the monography [3] for a full treatment of related questions) in considering the maximal monotone operator $v \mapsto A(v) := -\Delta v + |v|^{q-1}v$ seen as the subdifferential of the proper lower semi-continuous function

$$J_\Omega(v) = \left\{\begin{array}{ll} \displaystyle\int_\Omega \left(\frac{1}{2}|\nabla v|^2 + \frac{1}{q+1}|v|^{q+1}\right) dx & \text{if } v \in H^1_0(\Omega) \cap L^{q+1}(\Omega) \\ \infty & \text{if } v \notin H^1_0(\Omega) \cap L^{q+1}(\Omega). \end{array}\right. \tag{2.2}$$

In that case, the domain of $A = \partial J_\Omega$ is $D(A) := \{u \in H^1_0(\Omega) \cap L^{q+1}(\Omega) : \Delta u \in L^2(\Omega)\}$, and we endow $D_\Omega(-\Delta,)$ with the graph norm of the Laplacian in $H^1_0(\Omega)$

$$\|v\|_{D_\Omega(-\Delta)} = \left(\int_\Omega \left((\Delta v)^2 + |\nabla v|^2 + v^2\right) dx\right)^{1/2}.$$

Brezis' result is the following.

Theorem 2.1 *Given $u_0 \in L^2(\Omega)$ there exists a unique function $v \in L^2_{loc}(0,\infty; D_\Omega(-\Delta)) \cap C(0,\infty; H^1_0(\Omega) \cap L^{q+1}(\Omega))$ such that $\partial_t v \in L^2_{loc}(0,\infty; L^2(\Omega))$ satisfying*

$$\left\{ \begin{array}{rl} \partial_t v - \Delta v + |v|^{q-1}v = 0 & a.e. \text{ in } Q^\Omega_\infty \\ v(.,0) = u_0 & a.e. \text{ in } \Omega. \end{array} \right. \tag{2.3}$$

Furthermore the mapping $(t,u_0) \mapsto v(t,.)$ defines an order preserving contraction semigroup in $L^2(\Omega)$, denoted by $S^{\partial J_\Omega}(t)[u_0]$, and the following estimate holds

$$\|\partial_t v(t,.)\|_{L^2(\Omega)} \leq \frac{1}{t\sqrt{2}} \|u_0\|_{L^2(\Omega)}. \tag{2.4}$$

From this result, we have only to consider solutions of (2.1) with the above regularity.

Definition 2.2 *We denote by $\mathcal{I}(Q^\Omega_\infty)$ the set of positive functions $u \in L^2_{loc}(0,\infty; D_\Omega(-\Delta)) \cap C(0,\infty; H^1_0(\Omega) \cap L^{q+1}(\Omega))$ such that $\partial_t u \in L^2_{loc}(0,\infty; L^2(\Omega))$ satisfying*

$$\partial_t u - \Delta u + |u|^{q-1}u = 0 \tag{2.5}$$

in the semigroup sense, i. e.

$$\frac{du}{dt} + \partial J_\Omega(u) = 0 \quad a.e. \text{ in } (0,\infty). \tag{2.6}$$

If Ω is not bounded it is usefull to introduce another class which takes into account the Dirichlet condition on $\partial\Omega$: we assume that $\Omega^c \subset B_{R_0}$, denote by $\Omega_R = \Omega \cap B_R$ $(R \geq R_0)$ and by $\tilde{H}^1_0(\Omega_R)$ the closure in $H^1_0(\Omega_R)$ of the restrictions to Ω_R of functions in $C^\infty_0(\Omega)$, thus we endow $D_{\Omega_R}(-\Delta,)$ with the graph norm of the Laplacian in $\tilde{H}^1_0(\Omega_R)$

$$\|v\|_{D_{\Omega_R}(-\Delta)} = \left(\int_{\Omega_R} \left((\Delta v)^2 + |\nabla v|^2 + v^2 \right) dx \right)^{1/2}.$$

Definition 2.3 *If Ω is not bounded but $\Omega^c \subset B_{R_0}$, we denote by $\mathcal{I}(Q^{\Omega_{loc}}_\infty)$ the set of positive functions $u \in L^2_{loc}(Q^\Omega_\infty)$ such that, for any $R > R_0$, $u \in L^2_{loc}(0,\infty; D_{\Omega_R}(-\Delta)) \cap C(0,\infty; \tilde{H}^1_0(\Omega_R) \cap L^{q+1}(\Omega_R))$, $\partial_t u \in L^2_{loc}(0,\infty; L^2(\Omega_R))$ and u satisfies (2.5) in a. e. in Q^Ω_∞.*

Lemma 2.4 *If $u \in \mathcal{I}(Q^\Omega_\infty)$ or $\mathcal{I}(Q^{\Omega_{loc}}_\infty)$, its extension $\tilde{u}$ by zero outside Ω is a subsolution of (2.1) in $(0,\infty) \times \mathbb{R}^N$ such that $\tilde{u} \in C(0,\infty; H^1_0(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N))$ and $\partial_t \tilde{u} \in L^2_{loc}(0,\infty; L^2(\mathbb{R}^N))$.*

Proof. The proof being similar in the two cases, we assume Ω bounded. We first notice that $\tilde{u} \in C(0,\infty; H^1_0(\mathbb{R}^N))$ since $\|\tilde{u}\|_{H^1_0(\mathbb{R}^N)} = \|u\|_{H^1_0(\Omega)}$. For $\delta > 0$ we set

$$P_\delta(r) = \left\{ \begin{array}{ll} r - 3\delta/2 & \text{if } r \geq 2\delta \\ r^2/2\delta - r + \delta/2 & \text{if } \delta < r < 2\delta \\ 0 & \text{if } r \leq \delta \end{array} \right.$$

and denote by u_δ the extension of $P_\delta(u)$ by zero outside Q^Ω_∞. Since $u_{\delta\, t} = P'_\delta(u)\partial_t u$, then $u_{\delta\, t} \in L^2_{loc}(0,\infty; L^2(\mathbb{R}^N))$ and $\|u_{\delta\, t}\|_{L^2} \leq \|\partial_t u\|_{L^2}$. In the same way $\nabla u_\delta = P'_\delta(u)\nabla u$, thus $u_\delta \in L^2_{loc}(0,\infty; H^1_0(\mathbb{R}^N))$ and $\|u_\delta\|_{H^1_0} \leq \|u\|_{H^1_0}$. Finally $-\Delta u_\delta = -P'_\delta(u)\Delta u - P''_\delta(u)\,|\nabla u|^2$. Using the fact that $P'_\delta u^q \geq u^q_\delta$, we derive from (2.6)

$$\partial_t u_\delta - \Delta u_\delta + u^q_\delta \leq 0$$

in the sense that

$$\iint_{Q^{\mathbb{R}^N}_\infty} (\partial_t u_\delta \zeta + \nabla u_\delta . \nabla \zeta + u^q_\delta \zeta)\, dxdt \leq 0 \tag{2.7}$$

for all $\zeta \in C^\infty((0,\infty)\times\mathbb{R}^N)$, $\zeta \geq 0$. Actually, $C^\infty((0,\infty)\times\mathbb{R}^N)$ can be replaced by $L^2(\epsilon,\infty; H^1_0(\mathbb{R}^N)) \cap L^{q'}((\epsilon,\infty)\times\mathbb{R}^N)$. Letting $\delta \to 0$ and using Fatou's theorem implies that (2.7) holds with u_δ replaced by $\tilde{u}$. □

Lemma 2.5 *For any $u \in \mathcal{I}(Q^\Omega_\infty)$, there holds*

$$u(x,t) \leq \left(\frac{1}{(q-1)t}\right)^{1/(q-1)} := \phi_q(t) \quad \forall (x,t) \in Q^\Omega_\infty. \tag{2.8}$$

Proof. Let $\tau > 0$. Since the function $\phi_{q,\tau}$ defined by $\phi_{q,\tau}(t) = \phi_q(t-\tau)$ is a solution of

$$\phi'_{q,\tau} + \phi^q_{q,\tau} = 0$$

and $(u - \phi_{q,\tau})_+ \in C(0,\infty; H^1_0(\Omega))$, there holds

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u-\phi_{q,\tau})^2_+ dx + \iint_{Q^\Omega_\infty} \left(\nabla u . \nabla (u-\phi_{q,\tau})_+ + (u^q - \phi^q_{q,\tau})(u-\phi_{q,\tau})_+\right) dxdt = 0.$$

Thus $s \mapsto \|(u-\phi_{q,\tau})_+(s)\|_{L^2}$ is nonincreasing. By Lebesgue's theorem,

$$\lim_{s\downarrow\tau} \|(u-\phi_{q,\tau})_+(s)\|_{L^2} = 0,$$

thus $u(x,t) \leq \phi_{q,\tau}(t)$ a.e. in Ω. Letting $\tau \downarrow 0$ and using the continuity yields to (2.8). □

Theorem 2.6 *For any $q > 1$, the set $\mathcal{I}(Q^\Omega_\infty)$ admits a least upper bound $\underline{u}_\Omega$ for the order relation. If Ω is bounded, $\underline{u}_\Omega \in \mathcal{I}(Q^\Omega_\infty)$; if it is not the case, then $\underline{u}_\Omega \in \mathcal{I}(Q^{\Omega_{loc}}_\infty)$.*

Proof. Step 1- Construction of $\underline{u}_\Omega$ when Ω is bounded. For $k \in \mathbb{N}^*$ we consider the solution $v = v_k$ (in the sense of Theorem 2.1 with the corresponding maximal operator in $L^2(\Omega)$) of

$$\left\{\begin{array}{ll} \partial_t v - \Delta v + v^q = 0 & \text{in } \Omega\times(0,\infty) \\ v(x,t) = 0 & \text{in } \partial\Omega\times(0,\infty) \\ v(x,0) = k & \text{in } \Omega. \end{array}\right. \tag{2.9}$$

When $k \to \infty$, v_k increases and converges to some $\underline{u}_\Omega$. Because of (2.8) and the fact that Ω is bounded, $\underline{u}_\Omega(t,.) \in L^2(\Omega)$ for $t > 0$. It follows from the closedness of maximal monotone operators that $\underline{u}_\Omega \in L^2_{loc}(0,\infty;D_\Omega(-\Delta)) \cap C(0,\infty;H^1_0(\Omega)\cap L^{q+1}(\Omega))$, $\partial_t\underline{u}_\Omega \in L^2_{loc}(0,\infty;L^2(\Omega))$ and

$$\frac{d\underline{u}_\Omega}{dt} + \partial J_\Omega(\underline{u}_\Omega) = 0 \quad \text{a.e. in } (0,\infty). \tag{2.10}$$

Thus $\underline{u}_\Omega \in \mathcal{I}(Q^\Omega_\infty)$. For $\tau,\epsilon>0$, the function $t\mapsto \underline{u}_\Omega(x,t-\tau)+\epsilon$ is a supersolution of (2.1). Let $u\in\mathcal{I}(Q^\Omega_\infty)$; for $k>\phi_q(\tau)$, the function $(x,t)\mapsto (u(x,t)-\underline{u}_\Omega(x;t-\tau)-\epsilon)_+$ is a subsolution of (2.1) and belongs to $C(\tau,\infty;H^1_0(\Omega))$. Since it vanishes at $t=\tau$, it follows from Brezis' result that it is identically zero, thus $u(x,t)\leq \underline{u}_\Omega(x,t-\tau)+\epsilon$. Letting $\epsilon,\tau\downarrow 0$ implies the claim.

Step 2- Construction of $\underline{u}_\Omega$ when Ω is unbounded. We assume that $\partial\Omega\subset B_{R_0}$ and for $n>R_0$, we recall that $\Omega_n=\Omega\cap B_n$. For $k>0$, we denote by $\underline{u}_{\Omega_n}$ the solution obtained in Step 1. Then $\underline{u}_{\Omega_n} = \lim_{k\to\infty} v_{n,k}$ where $v_{n,k}$ is the solution, in the sense of maximal operators in Ω_n of

$$\left\{\begin{array}{l} \dfrac{dv_{n,k}}{dt} + \partial J_{\Omega_n}(v_{n,k}) = 0 \quad \text{a.e. in } (0,\infty) \\[2mm] \qquad\qquad v_{n,k}(0) = k. \end{array}\right. \tag{2.11}$$

It follows from Lemma 2.4 that the extension $\tilde v_{n,k}$ by 0 of $v_{n,k}$ in Ω_{n+1} is a subsolution for the equation satisfied by $v_{n+1,k}$, with a smaller initial data, therefore $\tilde v_{n,k}\leq v_{n+1,k}$. This implies $\tilde{\underline{u}}_{\Omega_n}\leq u_{\Omega_{n+1}}$. Thus we define $\underline{u}_\Omega=\lim_{n\to\infty}\underline{u}_{\Omega_n}$. It follows from Lemma 2.5 and standard regularity results for parabolic equations that $u=\underline{u}_\Omega$ satisfies

$$\partial_t u - \Delta u + u^q = 0 \tag{2.12}$$

in Q^Ω_∞. Multiplying

$$\frac{d\underline{u}_{\Omega_n}}{dt} + \partial J_{\Omega_n}(\underline{u}_{\Omega_n}) = 0 \tag{2.13}$$

by $\eta^2\underline{u}_{\Omega_n}$ where $\eta\in C^\infty_0(\mathbb{R}^N)$ and integrating over Ω_n, yields to

$$2^{-1}\frac{d}{dt}\int_{\Omega_n}\eta^2\underline{u}^2_{\Omega_n}dx + \int_{\Omega_n}\left(|\nabla\underline{u}_{\Omega_n}|^2+\underline{u}^{q+1}_{\Omega_n}\right)\eta^2dx + 2\int_{\Omega_n}\nabla\underline{u}_{\Omega_n}.\nabla\eta\,\eta\underline{u}_{\Omega_n}dx = 0.$$

Thus, by Young's inequality,

$$2^{-1}\frac{d}{dt}\int_{\Omega_n}\eta^2\underline{u}^2_{\Omega_n}dx + \int_{\Omega_n}\left(2^{-1}|\nabla\underline{u}_{\Omega_n}|^2+\underline{u}^{q+1}_{\Omega_n}\right)\eta^2dx \leq 2\int_{\Omega_n}|\nabla\eta|^2\underline{u}^2_{\Omega_n}dx.$$

If we assume that $0\leq\eta\leq 1$, $\eta=1$ on B_R ($R>R_0$) and $\eta=0$ on B^c_{2R}, we derive, for any $0<\tau<t$,

$$\begin{aligned} 2^{-1}\int_{\Omega_n}\underline{u}^2_{\Omega_n}(.,t)\eta^2dx + \int_\tau^t\int_{\Omega_n}\left(2^{-1}\left|\nabla\underline{u}_{\Omega_n}\right|^2+\underline{u}^{q+1}_{\Omega_n}\right)\eta^2dxds \\ \leq 2\int_\tau^t\int_{\Omega_n}\underline{u}^2_{\Omega_n}|\nabla\eta|^2dxds + 2^{-1}\int_{\Omega_n}\underline{u}^2_{\Omega_n}(.,\tau)\eta^2dx. \end{aligned} \tag{2.14}$$

From this follows, if $n>2R$,

$$2^{-1}\int_{\Omega\cap B_R}\underline{u}^2_{\Omega_n}(.,t)dx + \int_\tau^t\int_{\Omega\cap B_R}\left(2^{-1}\left|\nabla\underline{u}_{\Omega_n}\right|^2+\underline{u}^{q+1}_{\Omega_n}\right)dxds \leq CR^N(t+1)\tau^{-2/(q-1)}. \tag{2.15}$$

If we let $n \to \infty$ we derive by Fatou's lemma

$$2^{-1}\int_{\Omega\cap B_R} \underline{u}_\Omega^2(.,t)dx + \int_\tau^t\int_{\Omega\cap B_R}\left(2^{-1}\left|\nabla \underline{u}_\Omega\right|^2 + \underline{u}_\Omega^{q+1}\right)dxds \leq CR^N(t+1)\tau^{-2/(q-1)}. \tag{2.16}$$

For $\tau > 0$ fixed, we multiply (2.13) by $(t-\tau)\eta^2 d\underline{u}_{\Omega_n}/dt$, integrate on $(\tau,t)\times\Omega_n$ and get

$$\begin{aligned}(t-\tau)\int_{\Omega_n}\left|\frac{d\underline{u}_{\Omega_n}}{dt}\right|^2\eta^2 dx + \frac{d}{dt}(t-\tau)\int_{\Omega_n}\left(\frac{|\nabla \underline{u}_{\Omega_n}|^2}{2} + \frac{\underline{u}_{\Omega_n}^{q+1}}{q+1}\right)\eta^2 dx \\ = \int_{\Omega_n}\left(\frac{|\nabla \underline{u}_{\Omega_n}|^2}{2} + \frac{\underline{u}_{\Omega_n}^{q+1}}{q+1}\right)\eta^2 dx - 2(t-\tau)\int_{\Omega_n}\nabla \underline{u}_{\Omega_n}.\nabla\eta\frac{d\underline{u}_{\Omega_n}}{dt}\eta dx.\end{aligned}$$

Since

$$2(t-\tau)\left|\int_{\Omega_n}\nabla \underline{u}_{\Omega_n}.\nabla\eta\frac{d\underline{u}_{\Omega_n}}{dt}\eta dx\right| \leq \frac{(t-\tau)}{2}\int_{\Omega_n}\left|\frac{d\underline{u}_{\Omega_n}}{dt}\right|^2\eta^2 dx + 4(t-\tau)\int_{\Omega_n}\left|\nabla \underline{u}_{\Omega_n}\right|^2|\nabla\eta|^2 dx,$$

we get, in assuming again $n > 2R$,

$$\begin{aligned}2^{-1}\int_\tau^t\int_\Omega (s-\tau)\left|\frac{d\underline{u}_{\Omega_n}}{dt}\right|^2\eta^2 dxds + (t-\tau)\int_\Omega\left(\frac{|\nabla \underline{u}_{\Omega_n}|^2}{2} + \frac{\underline{u}_{\Omega_n}^{q+1}}{q+1}\right)\eta^2 dx \\ \leq 4\int_\tau^t(s-\tau)\int_\Omega\left|\nabla \underline{u}_{\Omega_n}\right|^2|\nabla\eta|^2 dxds,\end{aligned} \tag{2.17}$$

from which follows,

$$\begin{aligned}2^{-1}\int_\tau^t\int_{\Omega\cap B_R}(s-\tau)\left|\frac{d\underline{u}_{\Omega_n}}{dt}\right|^2 dxds + (t-\tau)\int_{\Omega\cap B_R}\left(\frac{|\nabla \underline{u}_{\Omega_n}|^2}{2} + \frac{\underline{u}_{\Omega_n}^{q+1}}{q+1}\right)dx \\ \leq 4\int_\tau^t(s-\tau)\int_{\Omega\cap B_{2R}}\left|\nabla \underline{u}_{\Omega_n}\right|^2 dxds.\end{aligned} \tag{2.18}$$

The right-hand side of (2.18) remains uniformly bounded by $8C(2R)^N(t-\tau)t\tau^{-2/(q-1)}$ from (2.15). Then

$$\begin{aligned}2^{-1}\int_\tau^t\int_{\Omega\cap B_R}(s-\tau)\left|\frac{d\underline{u}_{\Omega_n}}{dt}\right|^2 dxds + (t-\tau)\int_{\Omega\cap B_R}\left(\frac{|\nabla \underline{u}_{\Omega_n}|^2}{2} + \frac{\underline{u}_{\Omega_n}^{q+1}}{q+1}\right)dx \\ \leq 8C(2R)^N(t-\tau)t\tau^{-2/(q-1)}\end{aligned} \tag{2.19}$$

By Fatou's lemma the same estimate holds if $\underline{u}_{\Omega_n}$ is replaced by $\underline{u}_\Omega$. Notice also that this estimate implies that $\underline{u}_\Omega$ vanishes in the H_0^1-sense on $\partial\Omega$ since $\eta\underline{u}_\Omega \in H_0^1(\Omega)$ where the function $\eta \in C_0^\infty(\mathbb{R}^N)$ has value 1 in B_R and $\Omega^c \subset B_R$. Moreover estimates (2.16) and (2.19) imply that $\underline{u}_\Omega$ satisfies (2.12) a.e., and thus it belongs to $\mathcal{I}(Q_\infty^{\Omega_{loc}})$.

Step 3- Comparison. At end, assume $u \in \mathcal{I}(Q_\infty^\Omega)$. For $R > n_0$ let W_R be the maximal solution of

$$-\Delta W_R + W_R^q = 0 \quad \text{in } B_R. \tag{2.20}$$

Existence follows from Keller-Osserman's construction [5],[8], and the following scaling and blow-up estimates holds

$$W_R(x) = R^{-2/(q-1)} W_1(x/R), \tag{2.21}$$

and

$$W_R(x) = C_q (R - |x|)^{-2/(q-1)} (1 + \circ(1)) \text{ as } |x| \to R. \tag{2.22}$$

For $\tau > 0$ set $v(x,t) = u(x,t) - \underline{u}_\Omega(x, t-\tau) - W_R(x)$. Then v_+ is a subsolution. Since $v(.,\tau) \in L^2(\Omega)$, $\lim_{s \downarrow \tau} ||v_+(.,s)||_{L^2} = 0$. Because $\eta \underline{u}_\Omega \in H^1_0(\Omega)$ for η as above, $\eta v_+ \in H^1_0(\Omega)$. Next, $supp\, v_+ \subset \Omega \cap B_R$. Since $u, \underline{u}_\Omega$ are locally in H^1, we can always assume that their restrictions to $\partial B_R \times [0,T]$ are integrable for the corresponding Hausdorff measure. Therefore Green's formula is valid, which implies

$$-\int_\tau^t \int_{\Omega \cap B_R} \Delta v_+ dxdt = \int_\tau^t \int_{\Omega \cap B_R} |\nabla v_+|^2 dxdt \quad \forall t > \tau.$$

Therefore

$$\int_{\Omega \cap B_R} v_+^2(x,t)dx + \int_\tau^t \int_{\Omega \cap B_R} \left(|\nabla v_+|^2 + (u - (\underline{u}_\Omega(., t-\tau) + W_R)^q) v_+\right) dxdt \le \int_{\Omega \cap B_R} v_+^2(x,s)dx.$$

We let $s \downarrow \tau$ and get $v_+ = 0$, equivalently $u(x,t) \le \underline{u}_\Omega(x, t-\tau) + W_R(x)$. Then we let $R \to \infty$ and $\tau \to 0$ and obtain $u(x,t) \le \underline{u}_\Omega(x,t)$, which is the claim. $\square$

Corollary 2.7 *Assume $\Omega^1 \subset \Omega^2 \subset \mathbb{R}^N$ are open domains, then $\underline{u}_{\Omega^1} \le \underline{u}_{\Omega^2}$. Furthermore, if $\Omega = \cup \Omega^n$ where $\Omega^n \subset \Omega^{n+1}$, then*

$$\lim_{n \to \infty} \underline{u}_{\Omega^n} = \underline{u}_\Omega, \tag{2.23}$$

locally uniformly in Q^Ω_∞.

Proof. The first assertion follows from the proof of Theorem 2.6. It implies

$$\lim_{n \to \infty} \underline{u}_{\Omega^n} = u^*_\Omega \le \underline{u}_\Omega,$$

and u^*_Ω is a positive solution of (2.5) in Q^Ω_∞. There exists a sequence $\{u_{0,m}\} \subset L^2(\Omega)$ such that $S^{\partial J_\Omega}(t)[u_{0,m}] \uparrow \underline{u}_\Omega$ as $n \to \infty$, locally uniformly in Q^Ω_∞. Set $u_{0,m,n} = u_{0,m}\chi_{\Omega^n}$; since $u_{0,m,n} \to u_{0,m}$ in $L^2(\Omega)$ then $S^{\partial J_\Omega}(.)[u_{0,m,n}] \uparrow S^{\partial J_\Omega}(.)[u_{0,m}]$ in $L^\infty(0,\infty; L^2(\Omega))$. If $\tilde{v}_{m,n}$ is the extension of $v_{m,n} := S^{\partial J_{\Omega^n}}(.)[u_{0,m,n}]$ by zero outside $Q^{\Omega_n}_\infty$ it is a subsolution smaller than $S^{\partial J_\Omega}(.)[u_{0,m,n}]$ and $n \mapsto \tilde{v}_{m,n}$ is increasing; we denote by $\tilde{v}_m$ its limit as $n \to \infty$. Since for any $\zeta \in C^{2,1}_0([0,\infty) \times \Omega)$ we have, for n large enough and $s > 0$,

$$-\int_0^s \int_\Omega \left(\tilde{v}_{m,n} \left(\partial_t \zeta + \Delta \zeta\right)\right) dxdt = \int_\Omega u_{0,m,n} \zeta(x,0)dx - \int_\Omega \tilde{v}_{m,n}(x,s)\zeta(x,t)dx,$$

it follows

$$-\int_0^s \int_\Omega \left(\tilde{v}_m \left(\partial_t \zeta + \Delta \zeta\right)\right) dxdt = \int_\Omega u_{0,m} \zeta(x,0)dx - \int_\Omega \tilde{v}_m(x,s)\zeta(x,t)dx.$$

Clearly $\tilde{v}_m$ is a solution of (2.5) in $Q_\infty^{\Omega_n}$. Furthermore

$$\lim_{t\to 0} \tilde{v}_m(t,.) = u_{0,m} \quad \text{a.e. in } \Omega.$$

Because

$$\|\tilde{v}_m(t,.) - u_{0,m}\|_{L^2(\Omega)} \le 2\,\|u_{0,m}\|_{L^2(\Omega)}\,,$$

it follows from Lebesgue's theorem that $t \mapsto \tilde{v}_m(t,.)$ is continuous in $L^2(\Omega)$ at $t = 0$. Furthermore, for any $t > 0$ and $h \in (-t,t)$, we have from 2.4 ,

$$\begin{aligned} \|\tilde{v}_{m,n}(t+h,.) - \tilde{v}_{m,n}(t,.)\|_{L^2(\Omega^n)} &\le \frac{|h|}{t\sqrt{2}}\,\|u_{0,m,n}\|_{L^2(\Omega^n)} \\ \Longrightarrow \|\tilde{v}_m(t+h,.) - \tilde{v}_m(t,.)\|_{L^2(\Omega)} &\le \frac{|h|}{t\sqrt{2}}\,\|u_{0,m}\|_{L^2(\Omega)}\,. \end{aligned} \tag{2.24}$$

Thus $\tilde{v}_m \in C([0,\infty); L^2(\Omega)$. By the contraction principle, $\tilde{v}_m = S^{\partial J_\Omega}(t)[u_{0,m}]$ is the unique generalized solution to (2.3). Finally, there exists an increasing sequence $\{u_{0,m}\} \subset L^2(\Omega)$ such that for any $\epsilon > 0$, and $\tau > 0$,

$$0 < \underline{u}_\Omega - S^{\partial J_\Omega}(t)[u_{0,m}] \le \epsilon/2$$

on $[\tau,\infty) \times \Omega$. For any m, there exists n_m such that

$$0 < S^{\partial J_\Omega}(t)[u_{0,m}] - \tilde{v}_{m,n} \le \epsilon/2$$

Therefore

$$0 < \underline{u}_\Omega - \underline{u}_{\Omega^n} \le \epsilon,$$

on $[\tau,\infty) \times \Omega_n$. This implies (2.23). □

We can also construct a minimal solution with conditional initial blow-up in the following way. Assuming that $\Omega = \cup\Omega^m$ where Ω^m are smooth bounded domains and $\overline{\Omega^m} \subset \Omega^{m+1}$. We denote by u_m the solution of

$$\left\{ \begin{aligned} \partial_t u_m - \Delta u_m + |u_m|^{q-1}u_m = 0 &\quad \text{in } Q_\infty^{\Omega^m} \\ u_m = 0 &\quad \text{in } \partial\Omega^m \times (0,\infty) \\ \lim_{t\to 0} u_m(x,t) = \infty &\quad \text{locally uniformly on } \Omega^m. \end{aligned} \right. \tag{2.25}$$

Such a u_m is the increasing limit as $k \to \infty$ of the solutions $u_{m,k}$ of the same equation, with same boundary data and initial value equal to k. Since $\overline{\Omega}^m \subset \Omega^{m+1}$, $u_m < u_{m+1}$. We extend u_m by zero outside Ω^m and the limit of the sequence $\{u_m\}$, when $m \to \infty$ is a positive solution of (2.5) in Q_∞^Ω. We denote it by u_Ω. The next result is similar to Corollary 2.7, although the proof is much simpler.

Corollary 2.8 *Assume $\Omega^1 \subset \Omega^2 \subset \mathbb{R}^N$ are open domains, then $u_{\Omega^1} \le u_{\Omega^2}$. Furthermore, if $\Omega = \cup\Omega^n$ where $\Omega^n \subset \Omega^{n+1}$, then*

$$\lim_{n\to\infty} u_{\Omega^n} = u_\Omega, \tag{2.26}$$

locally uniformly in Q_∞^Ω.

Proposition 2.9 *There holds* $u_\Omega = \underline{u}_\Omega$.

Proof. For any $m, k > 0$, $\tilde{u}_{m,k}$, the extension of $u_{m,k}$ by zero in $Q_\infty^{\Omega^{m\,c}}$ is a subsolution, thus it is dominated by $\underline{u}_\Omega$. Letting successively $k \to \infty$ and $m \to \infty$ implies $u_\Omega \leq \underline{u}_\Omega$. In order to prove the reverse inequality, we consider an increasing sequence $\{u_\ell\} \subset \mathcal{I}(Q_\infty^\Omega)$ converging to $\underline{u}_\Omega$ locally uniformly in Q_∞^Ω. If Ω is bounded there exists a bounded sequence $\{u_{\ell,0,k}\}$ which converges to $u_\ell(.,0) = u_{\ell,0}$ in $L^2(\Omega)$ and $S^{\partial J_\Omega}(.)[u_{\ell,0,k}] \to S^{\partial J_\Omega}(.)[u_{\ell,0}]$ in $L^\infty(0,\infty; L^2(\Omega))$. Therefore

$$S^{\partial J_\Omega}(.)[u_{\ell,0,k}] \leq u_\Omega \Longrightarrow S^{\partial J_\Omega}(.)[u_{\ell,0}] \leq u_\Omega \Longrightarrow \underline{u}_\Omega \leq u_\Omega. \tag{2.27}$$

Next, if Ω is unbounded, $\Omega = \cup \Omega^n$, with $\Omega^n \subset \Omega^{n+1}$ are bounded, we have

$$\lim_{n\to\infty} \underline{u}_{\Omega^n} = \underline{u}_\Omega$$

and

$$\lim_{n\to\infty} u_{\Omega^n} = u_\Omega$$

by Corollary 2.7 and Corollary 2.8. Since $\underline{u}_{\Omega^n} = u_{\Omega^n}$ from the first part of the proof, the result follows. □

Remark. By construction u_Ω is dominated by any positive solution of (2.12) which satisfies the initial blow-up condition locally uniformly in Ω. Therefore, $u_\Omega = \underline{u}_\Omega$ is the *minimal solution* with initial blow-up.

If Ω has the minimal regularity which allows the Dirichlet problem to be solved by any continuous function g given on $\partial\Omega \times [0,\infty)$, we can consider another construction of the maximal solution of (2.1) in Q_∞^Ω. The needed assumption on $\partial\Omega$ is known as the *parabolic Wiener criterion* [9] (abr. PWC).

Definition 2.10 *If $\partial\Omega$ is compact and satisfies PWC, we denote by $\mathcal{J}_{Q_\infty^\Omega}$ the set of $v \in C((0,\infty) \times \overline{\Omega}) \cap C^{2,1}(Q_\infty^\Omega)$ satisfying (2.1).*

Theorem 2.11 *Assume $q > 1$ and Ω satisfies PWC. Then $\mathcal{J}_{Q_\infty^\Omega}$ admits a maximal element $\overline{u}_\Omega$.*

Proof. Step 1- Construction. We shall directly assume that Ω is unbounded, the bounded case being a simple adaptation of our construction. We suppose $\Omega^c \subset B_{R_0}$, and for $n > R_0$ set $\Omega_n = \Omega \cap B_n$. The construction of u_n is standard: for $k \in \mathbb{N}_*$ we denote by $v_k^* = v_{n,k}^*$ the solution of (2.9). Lemma 2.5 is valid for v_k^*. Notice that uniqueness follows from the maximum principle. When $k \to \infty$ the sequence $\{v_k\}$ increases and converges to a solution u_n of (2.12) in Q_{Ω_n}. Because the exterior boundary of Ω_n is smooth, the standard equicontinuity of the sequence of solutions applies, thus $u_n(x,t) = 0$ for all (x,t) s.t. $|x| = n$ and $t > 0$. In order to see that $u_n(x,t) = 0$ for all (x,t) s.t. $x \in \partial\Omega$ and $t > 0$, we see that $u_n(x,t) \leq \phi_\tau(x,t)$ on $(\tau,\infty) \times \Omega_n$, where

$$\left\{\begin{array}{ll} \partial_t\phi_\tau - \Delta\phi_\tau + \phi^{\tau\, q} = 0 & \text{in } Q_\infty^\Omega \\ \phi_\tau(x,\tau) = \phi_q(\tau) & \text{in } \Omega \\ \phi_\tau(x,t) = 0 & \text{in } \partial\Omega \times [\tau,\infty) \end{array}\right. \tag{2.28}$$

Such a solution exists because of PWC assumption. Since $v^*_{n,k}$ is an increasing function of n (provided the solution is extended by 0 outside Ω_n) and k, there holds $\tilde{u}_n \leq u_{n+1}$ in Ω^{n+1}. If we set

$$\overline{u}_\Omega = \lim_{n\to\infty} \tilde{u}_n,$$

then $\overline{u}_\Omega \leq \phi_\tau$ for any $\tau > 0$. Clearly $\overline{u}_\Omega$ is a solution of (2.12) in Q^Ω_∞. This implies that $\overline{u}_\Omega$ is continuous up to $\partial\Omega \times (0, \infty)$, with zero boundary value. Thus it belongs to $\mathcal{J}_{Q^\Omega_\infty}$.

Step 2- Comparison. In order to compare $\overline{u}_\Omega$ to any other $u \in \mathcal{J}_{Q^\Omega_\infty}$, for $R > R_0$ we set $v_{R,\tau}(x, t) = \overline{u}_\Omega(x, t - \tau) + W_R(x)$, where W_R is the maximal solution of (2.20) in B_R. The function $(u - v_{R,\tau})_+$ is a subsolution of (2.12) in $\Omega \cap B_R \times (\tau, \infty)$. It vanishes in a neighborhood on $\partial(\Omega \cap B_R) \times (\tau, \infty)$ and of $\Omega \cap B_R \times \{\tau\}$. Thus it is identically zero. If we let $R \to \infty$ in the inequality $u \leq v_{R,\tau}$ and $\tau \to 0$, we derive $u \leq \overline{u}_\Omega$, which is the claim. $\square$

Proposition 2.12 *Under the assumptions of Theorem 2.11, $\overline{u}_\Omega \in \mathcal{I}(Q^\Omega_\infty)$ if Ω is bounded and $\overline{u}_\Omega \in \mathcal{I}(Q^{\Omega_{loc}}_\infty)$ if Ω is not bounded.*

Proof. Case 1: Ω bounded. Let Ω^n be a sequence of smooth domains such that

$$\Omega^n \subset \overline{\Omega^n} \subset \Omega^{n+1} \subset \Omega$$

and $\cup_n \Omega^n = \Omega$. For $\tau > 0$, let $u_{n,\tau}$ be the solution of

$$\begin{cases} \partial_t u_{n,\tau} - \Delta u_{n,\tau} + u^q_{n,\tau} = 0 \quad \text{in } \Omega^n \times (\tau, \infty) \\ u_{n,\tau}(., \tau) = \overline{u}_\Omega(., \tau) \quad \text{in } \Omega^n \\ u_{n,\tau}(x, t) = 0 \quad \text{in } \partial\Omega^n \times [\tau, \infty) \end{cases} \tag{2.29}$$

Because $\overline{u}_\Omega(., \tau) \in C^2(\overline{\Omega}^n)$, $u_{n,\tau} \in C^{2,1}(\overline{\Omega}^n \times [\tau, \infty))$. By the maximum principle,

$$0 \leq \overline{u}_\Omega(., t) - u_{n,\tau}(., t) \leq \max\{\overline{u}_\Omega(x, s) : (x, s) \in \partial\Omega^n \times [\tau, t]\} \tag{2.30}$$

for any $t > \tau$. Because $\overline{u}_\Omega$ vanishes on $\partial\Omega \times [\tau, t]$, we derive

$$\lim_{n\to\infty} \tilde{u}_{n,\tau} = \overline{u}_\Omega \tag{2.31}$$

uniformly on $\overline{\Omega} \times [\tau, t]$ for any $t \geq \tau$, where $\tilde{u}_{n,\tau}$ is the extension of $u_{n,\tau}$ by zero outside Ω_n. Applying (2.15) and (2.19) with $\eta = 1$ to $\tilde{u}_{n,\tau}$ in Ω yields to

$$2^{-1} \int_\Omega \tilde{u}^2_{n,\tau}(., t)dx + \int_\tau^t \int_\Omega \left(|\nabla \tilde{u}_{n,\tau}|^2 + \tilde{u}^{q+1}_{n,\tau}\right) dxds \leq C(t + 1)\tau^{-2/(q-1)}. \tag{2.32}$$

and

$$2^{-1} \int_\tau^t \int_\Omega (s - \tau)(\partial_s \tilde{u}_{n,\tau})^2 dxds + (t - \tau) \int_\Omega \left(\frac{|\nabla \tilde{u}_{n,\tau}|^2}{2} + \frac{\tilde{u}^{q+1}_{n,\tau}}{q + 1}\right)(t, .)dx \leq C(t - \tau)t\tau^{-2/(q-1)}. \tag{2.33}$$

Letting $n \to \infty$ and using (2.31) yields to

$$2^{-1}\int_{\Omega}\overline{u}_{\Omega}^{2}(.,t)dx+\int_{\tau}^{t}\int_{\Omega}\left(|\nabla\overline{u}_{\Omega}|^{2}+\overline{u}_{\Omega}^{q+1}\right)dxds\leq C(t+1)\tau^{-2/(q-1)}. \tag{2.34}$$

and

$$2^{-1}\int_{\tau}^{t}\int_{\Omega}(s-\tau)(\partial_{s}\overline{u}_{\Omega})^{2}dxds+(t-\tau)\int_{\Omega}\left(\frac{|\nabla\overline{u}_{\Omega}|^{2}}{2}+\frac{\overline{u}_{\Omega}^{q+1}}{q+1}\right)(t,.)dx\leq C(t-\tau)t\tau^{-2/(q-1)}. \tag{2.35}$$

Since $L^2(\tau,t;H_0^1(\Omega))$ is a closed subspace of $L^2(\tau,t;H^1(\Omega))$, for any $0<\tau<t$, $\overline{u}_\Omega \in L^2_{loc}(0,\infty;H_0^1(\Omega))$. Furthermore $\partial_s\overline{u}_\Omega \in L^2_{loc}(0,\infty;L^2(\Omega))$. Because $\overline{u}_\Omega$ satisfies (2.12), it implies $\overline{u}_\Omega \in \mathcal{I}(Q_\infty^\Omega)$.

Case 2: Ω unbounded. We assume that $\Omega^c \subset B_{R_0}$. We consider a sequence of smooth unbounded domains $\{\Omega^n\} \subset \Omega$ $(n>1)$ such that $\sup\{\operatorname{dist}(x,\Omega^c) : x \in \partial\Omega^n\} < 1/n$ as $n \to \infty$, thus $\cup_n \Omega^n = \Omega$. For $m > R_0$ we set $\Omega^n_m = \Omega^n \cap B_m$. Therefore $\Omega^n_m \subset \overline{\Omega^n_m} \subset \Omega^{n+1}_{m+1}$ and $\cup_{n,m}\Omega^n_m = \Omega$. For $\tau > 0$, let $u = u_{m,n,\tau}$ be the solution of

$$\left\{\begin{aligned} &\partial_t u-\Delta u+u^q=0 \quad \text{in } \Omega^n_m\times(\tau,\infty)\\ &u(.,\tau)=\overline{u}_\Omega(.,\tau) \quad \text{in } \Omega^n_m\\ &u(x,t)=0 \quad \text{in } \partial\Omega^n\times[\tau,\infty)\\ &u(.,\tau)=\overline{u}_\Omega(.,\tau) \quad \text{in } \partial B_m\times(\tau,\infty).\end{aligned}\right. \tag{2.36}$$

By the maximum principle,

$$0\leq\overline{u}_\Omega(.,t)-u_{m,n,\tau}(.,t)\leq\max\{\overline{u}_\Omega(x,s):(x,s)\in\partial\Omega^n\times[\tau,t]\}\to 0, \tag{2.37}$$

as $n \to 0$. Next we extend $u_{m,n,\tau}$ by zero in $\Omega\setminus\Omega_n$ and apply (2.15)-(2.19) with η as in Theorem 2.6 and $m > 2R$. We get, with $\Omega_R = \Omega \cap B_R$,

$$2^{-1}\int_{\Omega_R}u_{m,n,\tau}^{2}(.,t)dx+\int_{\tau}^{t}\int_{\Omega_R}\left(2^{-1}|\nabla u_{m,n,\tau}|^{2}+u_{m,n,\tau}^{q+1}\right)dxds\leq CR^{N}(t+1)\tau^{-2/(q-1)}, \tag{2.38}$$

and

$$\begin{aligned}2^{-1}\int_{\tau}^{t}\int_{\Omega_R}(s-\tau)\left|\frac{du_{m,n,\tau}}{dt}\right|^{2}dxds+(t-\tau)\int_{\Omega_R}\left(\frac{|\nabla u_{m,n,\tau}|^{2}}{2}+\frac{u_{m,n,\tau}^{q+1}}{q+1}\right)dx\\ \leq 8C(2R)^{N}(t-\tau)t\tau^{-2/(q-1)}\end{aligned} \tag{2.39}$$

We let successively $m \to \infty$ and $n \to \infty$ and derive by Fatou's lemma and (2.37) that inequalities (2.38) and (2.39) still hold with $\overline{u}_\Omega$ instead of $u_{m,n,\tau}$. If we denote by $\tilde{H}^1_0(\Omega_R)$ the closure of the space of $C^\infty(\overline{\Omega_R})$ functions which vanish in a neighborhood on $\partial\Omega$, then (2.38) is an estimate in $L^2(\tau,t;\tilde{H}^1_0(\Omega_R))$ which is a closed subspace of $L^2(\tau,t;H^1(\Omega_R))$. Therefore $\overline{u}_\Omega \in L^2_{loc}(0,\infty;\tilde{H}^1_0(\Omega_R))$. Using (2.39) and equation (2.12) we conclude that $\overline{u}_\Omega \in \mathcal{I}(Q_\infty^{\Omega_{loc}})$. □

We end this section with a comparison result between $\underline{u}_\Omega$ and $\overline{u}_\Omega$.

Theorem 2.13 *Assume $q > 1$ and Ω satisfies PWC. Then $\underline{u}_\Omega = \overline{u}_\Omega$.*

Proof. By Proposition 2.9 and Theorem 2.11-Step 2, $\underline{u}_\Omega \leq \overline{u}_\Omega$. If Ω is bounded, we can compare $\underline{u}_\Omega(.,.)$ and $\overline{u}_\Omega(.+\tau,.)$ on $\Omega \times (0,\infty)$. Since $\underline{u}_\Omega$, the least upper bound of $\mathcal{I}(Q^\Omega_\infty)$ belongs to $\mathcal{I}(Q^\Omega_\infty)$, and $\overline{u}_\Omega(.+\tau,.) \in \mathcal{I}(Q^\Omega_\infty)$ we derive $\overline{u}_\Omega(.+\tau,.) \leq \underline{u}_\Omega(.,.)$, from which follows $\overline{u}_\Omega \leq \underline{u}_\Omega$. Next, if Ω is not bounded, we can proceed as in the proof of Theorem 2.6 by comparing $\underline{u}_\Omega(.,.) + W_R$ and $\overline{u}_\Omega(.+\tau,.)$ on $\Omega_R \times (0,\infty)$, where W_R is defined in (2.20). Because $(\overline{u}_\Omega(.+\tau,.) - \underline{u}_\Omega(.,.) - W_R(.))_+$ is a subsolution of (2.12) in $Q^{\Omega_R}_\infty$ which vanishes at $t = 0$ and near $\partial\Omega_R \times (0,\infty)$; it follows $\overline{u}_\Omega(.+\tau,.) \leq \underline{u}_\Omega(.,.) + W_R(.)$. Letting $R \to \infty$ and $\tau \to 0$ completes the proof. □

3 Uniqueness of large solutions

Definition 3.1 *Let $q > 1$ and $\Omega \subset \mathbb{R}^N$ be any domain. A positive function $u \in C^{2,1}(Q^\Omega_\infty)$ of (2.12) is a large initial solution if it satisfies*

$$\lim_{t\to 0} u(x,t) = \infty \quad \forall x \in \Omega, \tag{3.40}$$

uniformly on any compact subset of Ω.

We start with the following lemma

Lemma 3.2 *Assume $u \in C^{2,1}(Q^\Omega_\infty)$ is a large solution of (2.12), then for any open subset G such that $\overline{G} \subset \Omega$, there holds*

$$\lim_{t\to 0} t^{1/(q-1)} u(x,t) = c_q := \left(\frac{1}{q-1}\right)^{1/(q-1)} \quad \text{uniformly in } G. \tag{3.41}$$

Proof. By compactness, it is sufficient to prove the result when $G = B_\rho$ and $\overline{B}_\rho \subset B_{\rho'} \subset \Omega$. Let $\tau > 0$; by comparison, $u(x,t) \geq u_{B_{\rho'}}(x,t+\tau)$ for any $(x,t) \in Q^\Omega_\infty$. Letting $\tau \to 0$ yields to $u \geq u_{B_{\rho'}}$. Next for $\tau > 0$,

$$\phi_q(t+\tau) \leq u_{B_{\rho'}}(x,t) + u_{B^c_{\rho'}}(x,t) + W_R(x) \quad \forall (x,t) \in Q^{\mathbb{R}^N}_\infty.$$

Similarly

$$\max\{u_{B_{\rho'}}(x,t+\tau), u_{B^c_{\rho'}}(x,t+\tau)\} \leq \phi_q(t) + W_R(x) \quad \forall (x,t) \in Q^{\mathbb{R}^N}_\infty.$$

Letting $R \to \infty$ and $\tau \to 0$,

$$\max\{u_{B_{\rho'}}, u_{B^c_{\rho'}}\} \leq \phi_q \leq u_{B_{\rho'}} + u_{B^c_{\rho'}} \quad \text{in } Q^{\mathbb{R}^N}_\infty.$$

For symmetry reasons, $x \mapsto u_{B^c_{\rho'}}(x,t)$ is radially increasing for any $t > 0$, thus, for any $\rho < \rho'$ and $T > 0$, there exists $C_{\rho,T} > 0$ such that

$$u_{B^c_{\rho'}}(x,t) \leq C_{\rho,T} \quad \forall (x,t) \in B_\rho \times [0,T].$$

Therefore

$$\lim_{t\to 0} t^{1/(q-1)} u_{B_{\rho'}}(x,t) = c_q \quad \text{uniformly on } B_\rho.$$

Because

$$u_{B_{\rho'}}(x,t) \le u(x,t) \le \phi_q(t) \quad \forall (x,t) \in Q_\infty^\Omega,$$

(3.41) follows. □

As an immediate consequence of Lemma 3.2 and (2.23), we obtain

Proposition 3.3 *Assume $q > 1$ and $\partial\Omega$ is compact. Then u_Ω is a large solution.*

We start with the following uniqueness result

Proposition 3.4 *Assume $q > 1$, Ω satisfies PWC, $\partial\Omega$ is bounded, and either Ω or Ω^c is strictly starshaped with respect to some point. Then $\overline{u}_\Omega$ is the unique large solution belonging to $\mathcal{J}(Q_\infty^\Omega)$.*

Proof. Without loss of generality, we can suppose that either Ω or Ω^c is strictly starshaped with respect to 0. By Theorem 2.11, $\overline{u}_\Omega$ exists and, by (2.23) and Lemma 3.2, it is a large solution. Let $u \in \mathcal{J}(Q_\infty^\Omega)$ be another large solution. Clearly $u \le \overline{u}_\Omega$. If Ω is starshaped, then for $k > 1$, the function $u_k(x,t) := k^{2/(q-1)} u(kx, k^2 t)$ is a solution in Q_{Ω_k}, with $\Omega_k := k^{-1}\Omega$. Clearly it is a large solution and it belongs to $\mathcal{J}(Q_{\Omega_k})$. For $\tau \in (0,1)$, set $u_{k,\tau}(x,t) = u_k(x, t-\tau)$. Because $\partial\Omega$ is compact,

$$\lim_{k\downarrow 1} d_H(\partial\Omega, \partial\Omega_k) = 0,$$

where d_H denotes the Hausdorff distance between compact sets. By assumption $\overline{u}_\Omega \in C([\tau,\infty)\times\overline{\Omega})$ vanishes on $[\tau,\infty)\times\partial\Omega$, thus, for any $\epsilon > 0$, there exists $k_0 > 1$ such that for any

$$k \in (1, k_0] \Longrightarrow \sup\{\overline{u}_\Omega(x,t) : (x,t) \in [\tau, 1]\times\partial\Omega_k\} \le \epsilon.$$

Since $u_{k,\tau} + \epsilon$ is a super solution in Q_{Ω_k} which dominates $\overline{u}_\Omega$ on $[\tau,1]\times\partial\Omega_k$ and at $t = \tau$, it follows that $u_{k,\tau} + \epsilon \ge \overline{u}_\Omega$ in $(\tau,1]\times\Omega_k$. Letting successively $k \to 1$, $\tau \to 0$ and using the fact that ϵ is arbitrary, yields to $u \ge \overline{u}_\Omega$ in $(0,1]\times\Omega$ and thus in Q_∞^Ω. If Ω^c is starshaped, then the same construction holds provided we take $k < 1$ and use the fact that, for $R > 0$ large enough, $u_{k,\tau} + \epsilon + W_R$ is a super solution in $Q_{\Omega_k \cap B_R}$ which dominates $\overline{u}_\Omega$ on $[\tau,1]\times\partial\Omega_k\cap B_R$ and at $t = \tau$. Letting successively $R \to \infty$, $k \to 1$, $\tau \to 0$ and $\epsilon \to 0$ yields to $u \ge \overline{u}_\Omega$ □

As a consequence of Section 2, we have the more complete uniqueness theorem

Theorem 3.5 *Assume $q > 1$, $\Omega \subset \mathbb{R}^N$ is a domain with a bounded boundary $\partial\Omega$ satisfying PWC. Then for any $f \in C(\partial\Omega\times[0,\infty))$, $f \ge 0$, there exists a unique positive function $u = \overline{u}_{\Omega,f} \in C(\overline{\Omega}\times(0,\infty)) \cap C^{2,1}(Q_\infty^\Omega)$ satisfying*

$$\begin{cases} \partial_t u - \Delta u + |u|^{q-1}u = 0 & \text{in } Q_\infty^\Omega \\ u = f & \text{in } \partial\Omega\times(0,\infty) \\ \lim_{t\to 0} u(x,t) = \infty & \text{locally uniformly on } \Omega. \end{cases} \tag{3.42}$$

Proof. Step 1: Existence. It is a simple adaptation of the proof of Theorem 2.11. For $k, \tau > 0$, we denote by $u = u_{k,\tau,f}$ the solution of

$$\left\{\begin{array}{rl} \partial_t u - \Delta u + |u|^{q-1}u = 0 & \text{in } \Omega \times (\tau, \infty) \\ u = f & \text{in } \partial\Omega \times (\tau, \infty) \\ u(x,\tau) = k & \text{on } \Omega. \end{array}\right. \tag{3.43}$$

Notice that $u_{k,\tau,f}$ is bounded from above by $\overline{u}_\Omega(.,. - \tau) + v_{f,\tau}$, where $v_{f,\tau} = v$ solves

$$\left\{\begin{array}{rl} \partial_t v - \Delta v + |v|^{q-1}v = 0 & \text{in } \Omega \times (\tau, \infty) \\ v = f & \text{in } \partial\Omega \times (\tau, \infty) \\ v(x,\tau) = 0 & \text{on } \Omega. \end{array}\right. \tag{3.44}$$

If we let $k \to \infty$ we obtain a solution $u_{\infty,\tau,f}$ of the same problem except that the condition at $t = \tau$ becomes $\lim_{t\to\tau} u(x,t) = \infty$, locally uniformly for $x \in \Omega$. Clearly $u_{\infty,\tau,f}$ dominates in $\Omega \times (\tau,\infty)$ the restriction to this set of any $u \in C(\overline{\Omega} \times \infty)) \cap C^{2,1}(Q^\Omega_\infty)$ solution of (3.42), in particular $\overline{u}_\Omega$. Therefore $u_{\infty,\tau,f} \geq u_{\infty,\tau',f}$ in $\Omega \times (\tau,\infty)$ for any $0 < \tau' < \tau$. When $\tau \to 0$, $u_{\infty,\tau,f}$ converges to a function $\overline{u}_f$ which satisfies the lateral boundary condition $\overline{u}_{\Omega,f} = f$. Therefore $\overline{u}_{\Omega,f}$ satisfies (3.42).

Step 2: Uniqueness. Assume that there exists another positive function $u := u_f \in C(\overline{\Omega} \times (0,\infty)) \cap C^{2,1}(Q^\Omega_\infty)$ solution of (3.42). Then $u_f < \overline{u}_{\Omega,f}$. For $\tau > 0$, consider the solution $v := v_\tau$ of

$$\left\{\begin{array}{rl} \partial_t v - \Delta v + |v|^{q-1}v = 0 & \text{in } \Omega \times (\tau, \infty) \\ v = 0 & \text{in } \partial\Omega \times (\tau, \infty) \\ v(x,\tau) = u_f(x,\tau) & \text{on } \Omega. \end{array}\right. \tag{3.45}$$

Then $v_\tau \leq u_f$ in $\Omega \times (\tau,\infty)$. In the same way, we construct a solution $v : \tilde{v}_\tau$ of the same problem (3.45) except that the condition at $t = \tau$ is now $v(x,\tau) = \overline{u}_{\Omega,f}(x,\tau)$ for all $x \in \Omega$. Furthermore $v_\tau \leq \tilde{v}_\tau \leq \overline{u}_{\Omega,f}$. Next we adapt a method introduced in [6], [7] in a different context. We denote

$$Z_f = \overline{u}_{\Omega,f} - u_f \quad \text{and} \quad Z_{0,\tau} = \tilde{v}_\tau - v_\tau, \tag{3.46}$$

and, for $(r,s) \in \mathbb{R}^2_+$,

$$h(r,s) = \left\{\begin{array}{ll} \dfrac{r^q - s^q}{r - s} & \text{if } r \neq s \\ 0 & \text{if } r = s. \end{array}\right.$$

Since $r \mapsto r^q$ is convex on $\mathbb{R}_+$, there holds

$$\left\{\begin{array}{l} r_0 \geq s_0,\, r_1 \geq s_1 \\ r_1 \geq r_0,\, s_1 \geq s_0 \end{array}\right. \Longrightarrow h(r_1,s_1) \geq h(r_0,s_0).$$

This implies

$$h(u_{\Omega,f}, u_f) \geq h(\tilde{v}_\tau, v_\tau) \quad \text{in } \Omega \times [\tau,\infty). \tag{3.47}$$

Next we write

$$\begin{aligned} 0 &= \partial_t(Z_f - Z_{0,\tau}) - \Delta(Z_f - Z_{0,\tau}) + \overline{u}^q_{\Omega,f} - u^q_f - (\tilde{v}^q_\tau - v^q_\tau) \\ &= \partial_t(Z_f - Z_{0,\tau}) - \Delta(Z_f - Z_{0,\tau}) + h(\overline{u}_{\Omega,f}, u_f)Z_f - h(\tilde{v}_\tau, v_\tau)Z_{0,\tau}. \end{aligned} \tag{3.48}$$

Combining (3.47), (3.48) with the positivity of Z_f and $Z_{0,\tau}$, we derive

$$\partial_t(Z_f - Z_{0,\tau}) - \Delta(Z_f - Z_{0,\tau}) + h(\overline{u}_{\Omega,f}, u_f)(Z_f - Z_{0,\tau}) \leq 0, \tag{3.49}$$

in $\Omega \times (\tau, \infty)$. On $\partial\Omega \times [\tau, \infty)$ there holds $Z_f - Z_{0,\tau} = f - f = 0$. Furthermore, at at $t = \tau$, $Z_f(x, \tau) - Z_{0,\tau}(x, \tau) = \overline{u}_{\Omega,f}(x, t) - u_f(x, \tau) - \overline{u}_{\Omega,f}(x, t) + u_f(x, \tau) = 0$. By the maximum principle, it follows $Z_f \leq Z_{0,\tau}$ in $\Omega \times [\tau, \infty)$. Since $\tau > \tau' > 0$ implies $v_\tau(x, \tau) = u_f(x, \tau) \geq v_{\tau'}(x, \tau)$ and $\tilde{v}_\tau(x, \tau) = \overline{u}_{\Omega,f}(x, \tau) \geq \tilde{v}_{\tau'}(x, \tau)$, the sequences $\{v_\tau\}$ and $\tilde{v}_\tau$ converge to some functions $\{v_0\}$ and $\tilde{v}_0$ which belong to $C(\overline{\Omega} \times (0, \infty)) \cap C^{2,1}(Q^\Omega_\infty)$ and satisfy (3.42) with $f = 0$ on $\partial\Omega \times (0, \infty)$. Furthermore

$$\overline{u}_{\Omega,f} - u_f \leq \tilde{v}_0 - v_0. \tag{3.50}$$

Since $\overline{u}_{\Omega,f} \geq \overline{u}_\Omega$, $\tilde{v}_0 \geq \overline{u}_\Omega$, which implies that $\tilde{v}_0 = \overline{u}_\Omega$ by the maximality of $\overline{u}_\Omega$. If Ω' is any smooth bounded open subset such that $\overline{\Omega}' \subset \Omega$ there holds by an easy approximation argument $v_0 \geq u_{\Omega'}$ in $\Omega' \times (0, \infty)$. Therefore $v_0 \geq u_\Omega = \underline{u}_\Omega = \overline{u}_\Omega$, by Proposition 2.9 and Theorem 2.13. Applying again Theorem 2.13 we derive that the right-hand side of (3.50) is zero, which yields to $\overline{u}_{\Omega,f} = u_f$ □

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